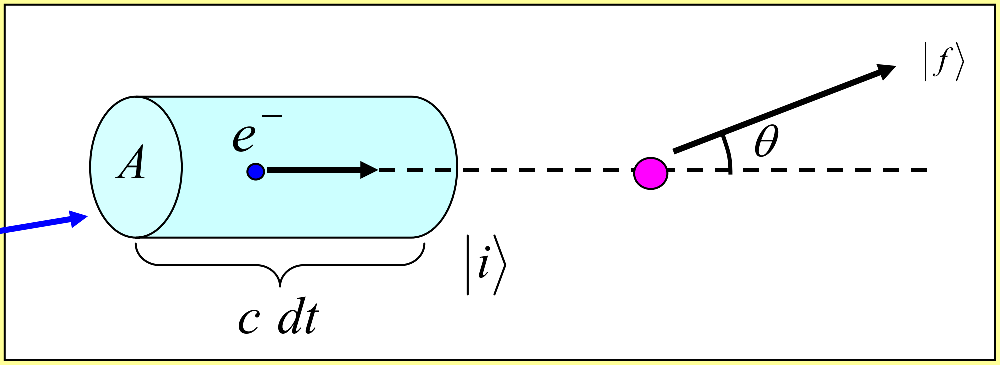
Recall:
A beam particle will scatter from the target particle into solid angle $d\Omega$ at (θ, ϕ) if it approaches within the corresponding area $d\sigma = (d\sigma/d\Omega)\, d\Omega$ centered on the target.

$V = A\, c\, dt$



- Electron (speed c) is in a plane wave state normalized in volume V as shown.
- Probability of scattering at angle θ is given by the ratio of areas: $P(\theta) = \frac{d\sigma(\theta)}{d\Omega} \frac{d\Omega}{A}$
- Transition rate λ_{if} = (electrons/Volume) x (Volume/time) x $P(\theta)$

$$\lambda_{if} = \left(\frac{1}{V}\right)\left(\frac{A\, c\, dt}{dt}\right)\left(\frac{d\sigma / d\Omega}{A}\right) d\Omega = \left(\frac{c}{V}\right)\left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

Recall from last time:

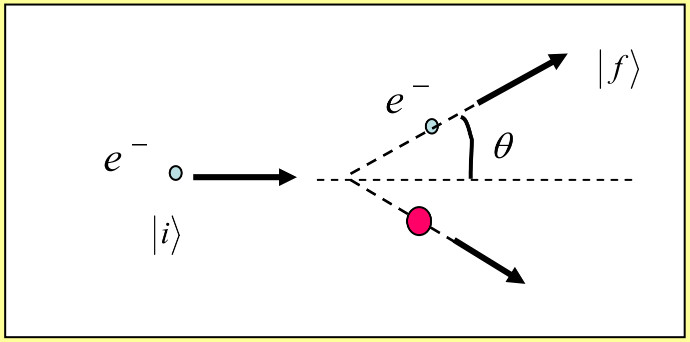
$$\lambda_{if} = \frac{2\pi}{\hbar} \left|M_{if}\right|^2 \rho_f$$

(transitions / sec)

$$\left\{ \begin{array}{l} \boxed{M_{if} = \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r} \\ \boxed{\rho_f = dn / dE_f} \end{array} \right.$$

And we just found that:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = \lambda_{if} \; \frac{V_n}{c \, d\Omega}$$

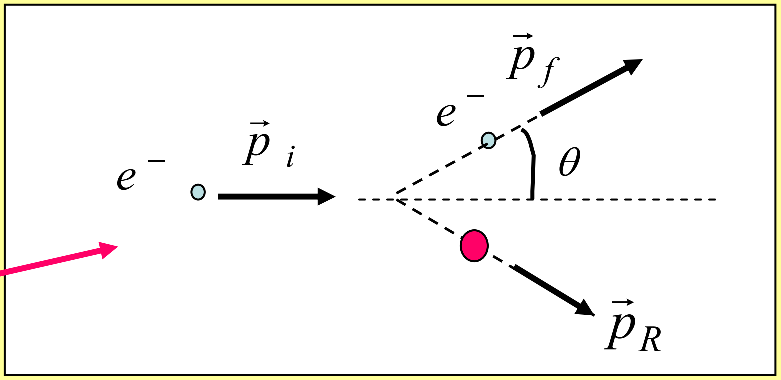


where V_n is the normalization volume for the plane wave electron states, and λ_{if} is the **transition rate** from the initial to final state, which we calculate using a standard result from quantum mechanics known as "Fermi's Golden Rule:"

We will first calculate the matrix element M_{if} and then the density of states ρ_f

$$M_{if} \equiv \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r$$

3 – momenta: p_i, p_f, p_R



Use plane wave states to represent the incoming and outgoing electrons, and

let $\boldsymbol{p_i = \hbar\, k_i}$, $\boldsymbol{p_f = \hbar\, k_f}$, $\boldsymbol{p_R = \hbar\, q}$, and **normalization volume = $\boldsymbol{V_n}$**

$$\psi_i(\vec{r}) = \frac{1}{\sqrt{V_n}} e^{i\vec{k}_i \bullet \vec{r}} \qquad \psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}} e^{i\vec{k}_f \bullet \vec{r}}$$

(Note: slight change of notation here from last class to make sure we don't miss any factors of ħ. The recoil momentum of the proton in MeV/c is p_R; the momentum transfer in fm $^{-1}$ is $q = p_R / \hbar$)

$$M_{if} \equiv \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r$$

$$= \frac{1}{V_n} \int e^{i(\vec{k}_i - \vec{k}_f) \cdot \vec{r}} \, V(\vec{r}) \, d^3 r = \frac{1}{V_n} \int e^{i \vec{q} \cdot \vec{r}} \, V(\vec{r}) \, d^3 r$$

Insight #1: RHS is the Fourier transform of the scattering potential $V(r)$, and it only depends on the momentum transfer q !

Next, proceed with caution:

$V(r)$ is the Coulomb potential of the **extended charge distribution** of the **target atom** that our electron is scattering from ...

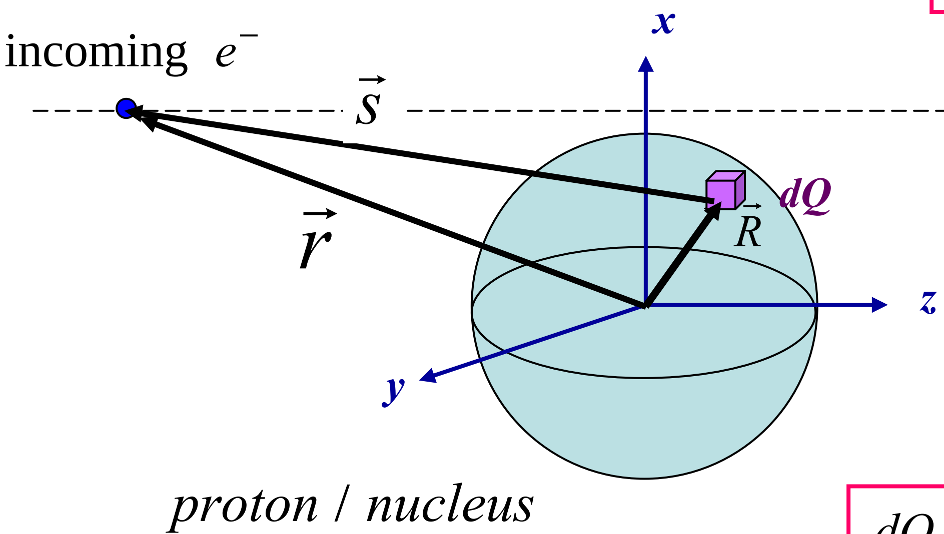
- **at large distances**, the atom is electrically neutral, so $V(r) \rightarrow 0$ **faster than 1/r**
- **at short distances**, we have to keep track of geometry carefully, **accounting for the details** of the proton (or nuclear) charge distribution....

Screened Coulomb potential:

$$V(r) = -\frac{Z e^2}{4\pi \varepsilon_o r} e^{-r/\alpha}$$

For the **atom**, where Z is the atomic number, and α is a distance scale of order Å, the atomic radius.

But the electron interacts with charge elements dQ inside the nucleus:



$$\vec{R} + \vec{s} = \vec{r}$$

$$dQ = +Ze\,\rho_e(\vec{R})\,d^3R$$

Bottom line:

$$V(r) = -e \int_{nucleus} \frac{dQ\; e^{-r/\alpha}}{4\pi \varepsilon_o s}$$

$$V(r) = -e \int_{nucleus} \frac{dQ\ e^{-r/\alpha}}{4\pi\ \varepsilon_o\ s}$$

$\vec{r}$ = electron coordinate

$\vec{R}$ = coordinate of dQ

$\vec{s}$ = displacement of electron from dQ

$$dQ = +Ze\,\rho_e(\vec{R})\,d^3R$$

$$\int dQ = +Ze$$

normalization: $[\rho] = m^{-3}$

substitute for dQ:

$$V(r) = -\frac{Z\,e^2}{4\pi\ \varepsilon_o} \int_{nucleus} \frac{\rho_e(R)\ e^{-r/\alpha}}{s}\ d^3R$$

Finally, for the matrix element:

$$M_{if} = \frac{1}{V_n} \int e^{i\,\vec{q}\cdot\vec{r}}\ V(\vec{r})\,d^3r = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\ \varepsilon_o}\right) \int_{all\ space} d^3r\ e^{i\vec{q}\cdot\vec{r}} \int_{nucleus} \frac{\rho_e(R)\ e^{-r/\alpha}}{s}\ d^3R$$

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\int_{all\ space} d^3r\ e^{i\vec{q}\bullet\vec{r}} \int_{nucleus} \frac{\rho_e(R)\ e^{-r/\alpha}}{s}\ d^3R$$

problem: r, R **and** s **in here!**

Solution:

1. inside the nucleus, where $\rho\,(\mathbf{R})$ differs from zero, $e^{-r/\alpha} \cong 1 \cong e^{-s/\alpha}$

   (where the screening factor really matters is at large r, and there r → s to an even better approximation!)

2. there is a one to one mapping between all electron positions $\boldsymbol{r}$ and all displacements from the charge element $\boldsymbol{dQ}$, so: $\int_{all\ space} d^3r = \int_{all\ space} d^3s$

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\iint e^{i\vec{q}\bullet\vec{r}}\ \rho_e(R)\ d^3R\ \frac{e^{-s/\alpha}}{s}\ d^3s$$

(this expression can be factored into 2 parts ...)

Finishing the integral for M_{if}:

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\iint e^{i\vec{q}\cdot\vec{r}}\ \rho_e(R)\ d^3R\ \frac{e^{-s/\alpha}}{s}\ d^3s$$

use the relation: $\vec{r} = \vec{R} + \vec{s}$ to simplify...

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\boxed{\int e^{i\vec{q}\cdot\vec{R}}\rho_e(R)\,d^3R} \times \boxed{\int e^{i\vec{q}\cdot\vec{s}}\ \frac{e^{-s/\alpha}}{s}\,d^3s}$$

Fourier transform of the nuclear charge density $\equiv$ ***F(q²)***

Exact integral: $\dfrac{4\pi}{q^2+\alpha^{-2}}$

(Eureka!) ☺ ⇒

$$\boxed{M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\left(\frac{4\pi}{q^2+\alpha^{-2}}\right)\ F(q^2)}$$

$$\boxed{M_{if} = \text{(constants)} \times \text{(exact integral)} \times \text{(Fourier transform of } \rho\text{(R))}}$$

$$M_{if} = \frac{1}{V_n}\left(-\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)\left(\frac{4\pi}{q^2+\alpha^{-2}}\right)F(q^2)$$

Consider:
$$\boxed{F(q^2) \equiv \int_{all\ space} e^{i\vec{q}\cdot\vec{r}}\,\rho_e(r)\,d^3r}$$

If the scattering object is a **point charge**, $\rho_e(r) = \delta^3(\vec{r})$, i.e. the normalized charge density is a **Dirac delta function**, with the property:

$$\int_{all\ space} \delta^3(\vec{r})\;d^3r \equiv 1 \quad \Longrightarrow \quad \boxed{F(q^2) = 1 \text{ for a point charge}}$$

(Really important result!)

N.B. for a delta function: $\int f(\vec{r})\,\delta^3(\vec{r})\,d^3r = f(0)$

from slide 2...

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{V_n}{c\,d\Omega}\,\left|M_{if}\right|^2 \rho_f$$

$$M_{if} = \int \psi_f^*\, V(\vec{r})\,\psi_i\, d^3 r$$

$$\rho_f = dn/dE_F$$

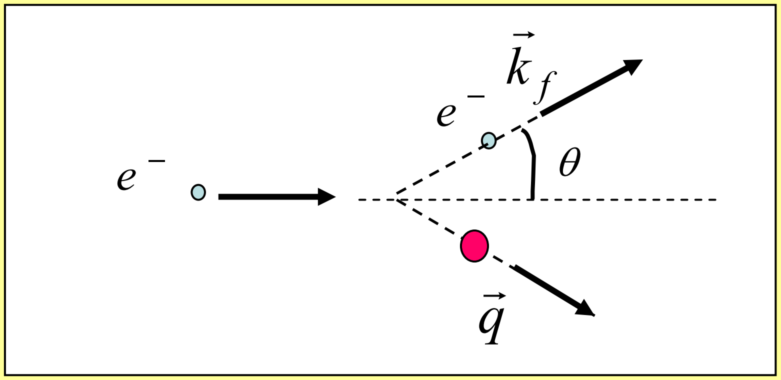
for the cross-section:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\,\frac{1}{c\,V_n}\left(\frac{Z e^2}{4\pi\,\varepsilon_o}\right)^2 \left(\frac{4\pi}{q^2+\alpha^{-2}}\right)^2 \left(F(q^2)\right)^2 \boxed{\frac{dn}{dE_F\, d\Omega}}$$

All we have left to calculate is the "density of states" factor, where E_F is the total energy in the final state when the electron scatters at angle θ, and this factor accounts for the number of ways it can do that.

$$\boxed{\frac{dn}{dE_F\, d\Omega}}$$

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{dp_f\, d\Omega}\left(\frac{dp_f}{dE_F}\right)_\theta$$



$$E_F = E' + E_R$$

(electron) *(recoil)*

$$E_F = (cp_f + mc^2) + (Mc^2 + K)$$

(being careful with the factor of c !)

$$\frac{dp_f}{dE_F} = \frac{W\, p_f}{Mc^3\, p_i} \approx \frac{1}{c}$$

$$dE_F \cong c\, dp_f$$

$$\frac{dn}{dE_F\, d\Omega} = \frac{dn}{c dp_f\, d\Omega}$$

This is useful because the momentum states are quantized – we have our electrons in a normalization volume, and we can "count the states" inside ...

Counting the scattered electron momentum states:

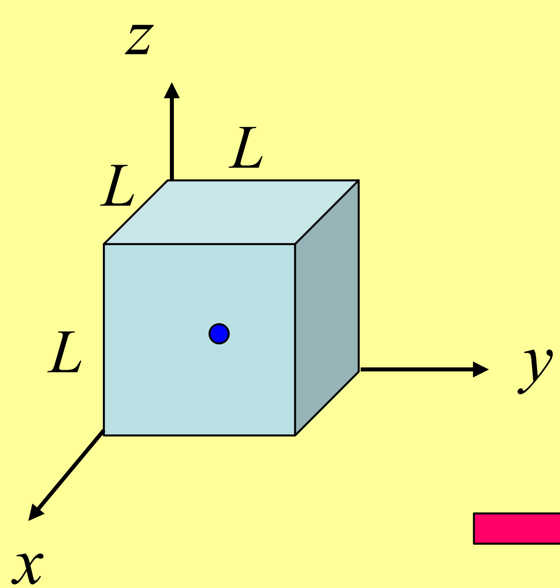
Recall the wave function:

$$\psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}} e^{i\vec{k}_f \bullet \vec{r}} \quad with \quad p_f = \hbar k_f$$

The normalization volume is arbitrary, but we have to be consistent ....

let $V_n = L^3$, i.e. the electron wave function is contained in a cubical box,

so its **wave function must be identically zero on all 6 faces of the cube.**



Since: $\vec{k}_f \bullet \vec{r} \equiv k_x x + k_y y + k_z z$

Then it follows that:

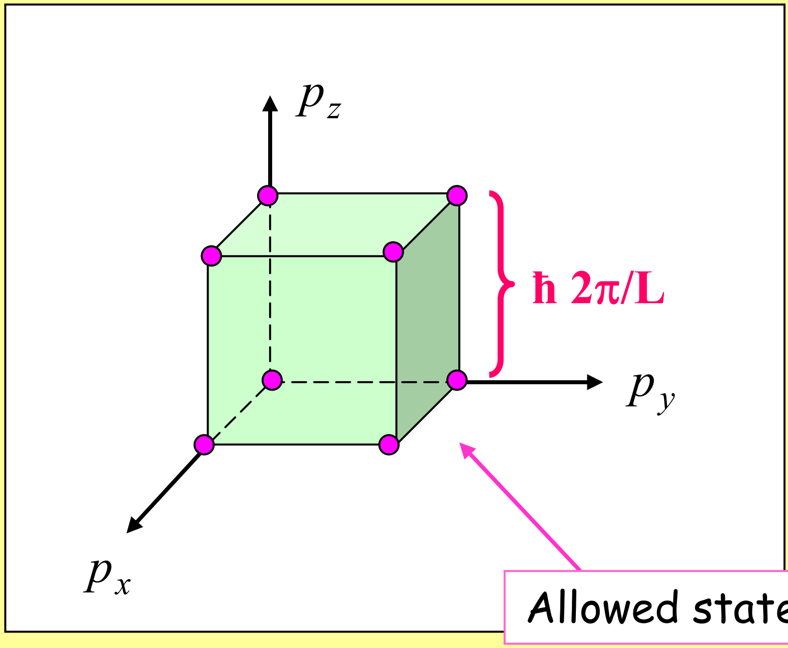
$$\psi_f(x,y,z) = \frac{1}{\sqrt{L^3}} e^{ik_x x} e^{ik_y y} e^{ik_z z}$$

$$\Rightarrow \quad k_x L = n_x \, 2\pi, \; etc. ...$$

So, momentum is quantized on a 3-d lattice:

$$\vec{p}_f = \hbar \vec{k}_f = \hbar \left(\frac{2\pi}{L}\right)\left(n_x \hat{i} + n_y \hat{j} + n_z \hat{k}\right)$$

$$n_x = \pm (1, 2, 3 \ldots) \quad etc.$$



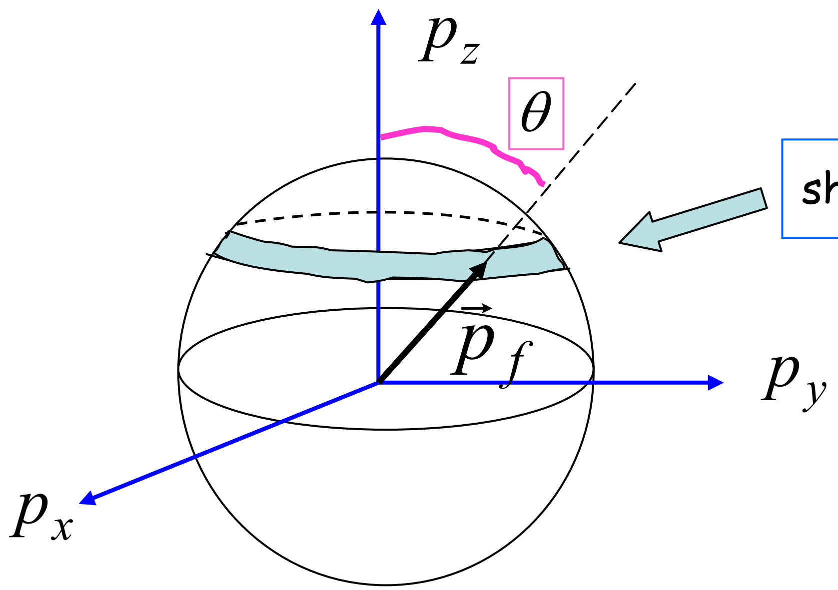
Allowed states are dots, 1 per cube of volume $\tau_p = (2\pi\hbar/L)^3$

For a relativistic electron beam, the quantum numbers n_x *etc.* are very large, but finite.

We use the quantization relation **not** to calculate the allowed momentum, but rather to calculate **the density of states!**

$$\frac{dn}{d\tau_p} = \frac{1 \text{ state}}{(2\pi\,\hbar / L)^3}$$

Finally, consider the scattered momentum into $d\Omega$ at θ :



shaded area: $dA_p = p_f^2 \, d\Omega$

number of momentum points in the shaded ring: $dn = \left(\frac{dn}{d\tau_p} \right) \times \left(dA_p \, dp_f \right)$

$$dn = \frac{V_n}{(2\pi \hbar)^3} \, p_f^2 \, dp_f \, d\Omega$$

End of the calculation:

$$dn = \frac{V_n}{(2\pi\,\hbar)^3}\; p_f^2\; dp_f\; d\Omega$$

We want the density of states factor:

$$\boxed{\frac{dn}{dE_F\; d\Omega} = \frac{dn}{c\,dp_f\; d\Omega} = \frac{V_n}{(2\pi\,\hbar)^3}\,\frac{p_f^2}{c}}$$

FINALLY, from slide 10:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{2\pi}{\hbar}\;\frac{1}{c\,\cancel{V_n}}\left(\frac{Z\,e^2}{4\pi\,\varepsilon_o}\right)^2\left(\frac{4\pi}{q^2+\alpha^{-2}}\right)^2\left(F(q^2)\right)^2\boxed{\left(\frac{\cancel{V_n}}{(2\pi\,\hbar)^3}\;\frac{p_f^2}{c}\right)}$$

$$= \left(\text{point charge cross-section}\right)\times\left(F(q^2)\right)^2$$

Result: Cross section for electron scattering from nuclear charge Z:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{4\,Z^2}{\hbar^2\,(\hbar c)^2}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2 \frac{p_f^2}{(q^2+\alpha^{-2})^2}\left(F(q^2)\right)^2$$

$$\cong \frac{4\,Z^2}{(\hbar c)^4}\left(\frac{e^2}{4\pi\,\varepsilon_o}\right)^2 \frac{(cp_f)^2}{q^4}\left(F(q^2)\right)^2$$

point charge cross-section: most notably, falls off as q^{-4} (units should be fm^2)

form factor squared (dimensionless)

Check units: $[\hbar c] = [e^2/4\pi\varepsilon_o] = \mathrm{MeV.fm};\; [cp] = \mathrm{MeV};\; [q] = \mathrm{fm}^{-1}$

$$\left[\frac{d\sigma}{d\Omega}\right] = \frac{1}{(\mathrm{MeV.fm})^4}(\mathrm{MeV.fm})^2\frac{(\mathrm{MeV})^2}{\mathrm{fm}^{-4}} = \mathrm{fm}^2$$